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LETTER TO THE EDITOR

Rotations and intelligence of coherent spin states

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Abstract. The minimum uncertainty relation which characterizes a coherent spin state is shown to be invariant under some special rotations and a model to interpret this result is given. 'Intelligence' of coherent states is discussed in the light of this interpretation.

A system of N two-level spins is in a typical coherent state (Radcliffe 1971, Arecchi *et al* 1972) if each spin of the system 'points' towards a common direction. More generally a coherent state can be defined directly in terms of the total spin operators $\boldsymbol{S}$, S_x, S_y, S_z: we may say that the system is in a coherent state if it has a well defined value of the total spin $|\boldsymbol{S}|$ ($S = 0, 1, \ldots, N/2$) and if the total spin 'points' towards a direction which is as well defined as it can be by the quantum nature of the system. If the selected direction is assumed to be the z axis, the coherent state satisfies the minimum uncertainty relation

$$\Delta S_x \Delta S_y = \tfrac{1}{2}|\langle S_z\rangle| \tag{1}$$

where $(\Delta S_x)^2 = \langle S_x^2\rangle - \langle S_x\rangle^2$ and where the mean values are taken on the coherent state itself.

In view of the relevance that coherent states may assume in many problems (Narducci *et al* 1973, Persico and Vetri 1975) and in connection with the work of Aragone *et al* (1974) concerning a particular class of coherent states which have been termed 'intelligent', it is useful to explore their properties of minimum uncertainty in differently rotated systems.

We shall inquire first if a coherent spin state 'pointing' towards the z direction satisfies the relation

$$\Delta S_{x'} \Delta S_{y'} = \tfrac{1}{2}|\langle S_{z'}\rangle|$$

where $S_{x'}$, $S_{y'}$, $S_{z'}$ are the spin components of the system in the x', y', z' directions which can be obtained from x, y, z by means of a rotation.

In terms of the matrix elements of the generic rotation matrix A and using the operator relation

$$(S_{x'} S_{y'} S_{z'})^\dagger = A(S_x S_y S_z)^\dagger$$

we find

$$(\Delta S_{x'})^2 = A_{11}^2(\Delta S_x)^2 + A_{12}^2(\Delta S_y)^2 = (A_{11}^2 + A_{12}^2)(\Delta S_x)^2$$

since our coherent state is an eigenstate of the S_z operator, and since $(\Delta S_x)^2 = (\Delta S_y)^2$.

In the same way we have

$$(\Delta S_{y'})^2 = (A_{21}^2 + A_{22}^2)(\Delta S_y)^2$$

and consequently

$$(\Delta S_{x'})^2(\Delta S_{y'})^2 = (A_{11}^2 + A_{12}^2)(A_{21}^2 + A_{22}^2)(\Delta S_x)^2(\Delta S_y)^2.$$

If we now express the matrix elements of A in terms of the Eulerian angles θ, ϕ and ψ (Goldstein 1959), we easily obtain

$$(\Delta S_{x'})^2(\Delta S_{y'})^2 = (\cos^2\theta + \sin^4\theta\sin^2\psi\cos^2\psi)(\Delta S_x)^2(\Delta S_y)^2 \tag{2}$$

and

$$|\langle S_{z'}\rangle|^2/4 = |\langle S_z\rangle|^2\cos^2\theta/4 \tag{3}$$

which are equal (see relation (1)) only for $\psi = 0$ or $\psi = \pi/2$. Thus we conclude that the minimum uncertainty relation is invariant only under rotations that leave x' or y' on the (xy) plane. That is to say, one has the minimum uncertainty relation only if at least one of the spin components ($S_{x'}$ or $S_{y'}$) remains in the plane perpendicular to the direction (z in our case) selected by the system.

These results can be visualized as follows. It is known that the spin momentum of a state having a well defined value of $|\boldsymbol{S}|$ and $S_z = S$, can be considered as spread out uniformly over a cone of height S along z and slant height equal to $[S(S+1)]^{1/2}$; hence the spin component on the (xy) plane is spread out over a circle of radius $S^{1/2}$. The mean value of the spin component S_x along x is zero and the variance

$$(\Delta S_x)^2 = \frac{1}{2\pi}\int_0^{2\pi} S\cos^2\alpha\,\mathrm{d}\alpha = S/2 = (\Delta S_y)^2.$$

In general the projection of the spin momentum on an $(x'y')$ plane will be an ellipse whose axes are functions of the Eulerian angles specifying the orientation of the $x'y'z'$ systems and the mean value of the spin component along z' is $\langle S_{z'}\rangle = \langle S_z\rangle\cos\theta$. As a consequence, the product of variances

$$(\Delta S_{x'})^2(\Delta S_{y'})^2 = (\cos^2\theta + \sin^4\theta\sin^2\psi\cos^2\psi)S^2/4$$

will be in general different from $\langle S_z\rangle^2/4$. However, if x' (or y') is chosen on the (xy) plane, one axis of the ellipse will be equal to the radius of the circle in the (xy) plane so that

$$(\Delta S_{x'}) = (\Delta S_x) = S/2 \qquad (\text{or } (\Delta S_{y'}) = (\Delta S_y) = S/2),$$

and it is immediate that

$$(\Delta S_{y'})^2 = (\Delta S_y)^2\cos^2\theta \qquad (\text{or } (\Delta S_{x'})^2 = (\Delta S_x)^2\cos^2\theta)$$

varies with θ as $\langle S_{z'}\rangle^2$ does. Only in this case the product of the variances is

$$(\Delta S_{x'})^2(\Delta S_{y'})^2 = \langle S_{z'}\rangle^2/4.$$

Consider now a coherent state 'pointing' towards a z' direction specified by the polar coordinates θ and φ. In this case the minimum uncertainty relation which characterizes the coherent state is obviously

$$(\Delta S_{x'})^2(\Delta S_{y'})^2 = \langle S_{z'}\rangle^2 \tag{4}$$

where x' and y' are two directions orthogonal to each other and to z'. Our previous results indicate, however, that the coherent state also satisfies the uncertainty relation

$$(\Delta S_{\hat{n}})^2(\Delta S_{\hat{n}_\perp})^2 = \langle S_z \rangle^2/4 \tag{5}$$

where $\hat{n} = (\sin\varphi, -\cos\varphi, 0)$ and $\hat{n}_\perp = (\cos\varphi, \sin\varphi, 0)$, because the spin component $S_{\hat{n}}$ lies in the $(x'y')$ plane perpendicular to the z' direction selected by the system and consequently

$$(\Delta S_{\hat{n}})^2 = (\Delta S_{x'})^2 = S/2, \qquad (\Delta S_{\hat{n}_\perp})^2 = (\Delta S_{y'})^2 \cos^2\theta \qquad \text{and} \qquad \langle S_z \rangle = S\cos\theta.$$

In actual cases the uncertainty relation (5) may be more practical than (4) to explore the coherence properties of a spin system.

We conclude that if the spin states which satisfy relation (1) are defined 'intelligent states' (Aragone *et al* 1974) any coherent or 'Radcliffe state' is an intelligent one if the z axis is chosen in the appropriate way. Moreover, the Radcliffe states which do not point in the z direction and are labelled 'intelligent' by Aragone *et al* (1974) are actually those oriented in such a way that the $\hat{n}$ axis is along x or y.

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